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HYDRODYNAMICAL VORTEX ON THE PLANE

1. INTRODUCTION

Hydrodynamics has always been a source of nonlinear equations integrable by the inverse scattering method. To mention a few, the Korteveg - de Vries, Kadomtsev - Petviashvili, Davey - Stewartson and Bendjamin - Ono equations all have hydrodynamical origin. As a rule, these equations can be obtained by direct application of procedures of the multiscale decomposition type to the equations of hydrodynamics. In other words, virtually all of them arise as a result of certain reductions of the initial system of nonlinear hydrodynamical equations, the latter having rather few known exact solutions (one should mention here the Landau solution for the submerged stream [1], and a rapidly developing approach to the integration of nonlinear equations based on the group theory [2]). It is natural then to search for completely integrable partial cases of this system, since such cases, if found, would allow regular procedures for constructing the solutions. This idea has apparently first been realized in [3], where a Lax pair for the Euler equation for ideal incompressible fluid was found. A more special case - the vortex equation - was studied in [4].

In this paper we consider vortical solutions of the $2D$ Euler equations, which have numerous applications, in particular, in geophysical hydrodynamics [5,6] and plasma physics [7]. The existence of a Lax pair (although, in a "weak" sense) for the problem allows to apply a range of tools from the theory of completely integrable systems.

Notice that the structure of the pair under consideration is rather special - since both equations of the pair are of the first order in the auxiliary function, no scattering problem occurs. This leads, in particular, to solutions containing arbitrary functions and certain other peculiarities.

The structure of the paper is as follows. In section 2 we give the necessary preliminaries from the hydrodynamics and the theory of vortices. Section 3 is devoted to solving a simplified vortex equation, section 4 to constructing exact 2D solutions of the vortex equation, both partial and more general, obtained by application of the Darboux transformation.

2. PRELIMINARIES

Let us consider the flow of an ideal incompressible two-dimensional fluid in a domain $\mathbb{D} \in \mathbb{R}^2$. Let $\vec{v} = \vec{v}(x, y) = (v_1, v_2)$ be the velocity field, p be the pressure, and ρ be the density of the fluid. Then the following system of equations on the velocity field holds [1]:

$$\mathrm{d}iv\, \vec{v} = 0, \tag{2.1}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v} = -\frac{\nabla p}{\rho}.$$

The first equation in (2.1) expresses the conservation of mass, the second one is the Euler equation.

This system should be provided with boundary conditions. It is natural to assume that if the domain $\mathbb{D}$ is bounded, then

$$(\vec{v}, \vec{n}) = 0 \quad \text{at} \quad (x, y) \in \partial\mathbb{D}, \tag{2.2}$$

where $\vec{n}$ is orthogonal to the boundary surface, which means that the fluid can not spill over the boundary, and that

$$\vec{v} \to 0 \quad \text{at} \quad \sqrt{x^2 + y^2} \to \infty, \tag{2.3}$$

if the domain is unbounded.

As is well known, the system (2.1) have an infinite series of the first integrals [8]. In particular, the quantities ($k = 1, 2.....$)

$$I_k = \int\int_{\mathbb{D}} (\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y})^k dx\, dy \tag{2.4}$$

are first integrals.

Using the identity

$$\frac{1}{2}\nabla v^2 = \vec{v} \wedge \text{rot}\, \vec{v} + (\vec{v}\nabla)\vec{v},$$

we find from the second equation in (2.1) that ($dw = dp/\rho$)

$$\frac{\partial \vec{v}}{\partial t} - \vec{v} \wedge \text{rot}\, \vec{v} = -\nabla(w + \frac{v^2}{2}).$$

Applying the rot operation to this equality, we obtain,

$$\frac{\partial \text{rot} \;\; \vec{v}}{\partial t} = \text{rot}\, [\, \vec{v} \wedge \text{rot} \;\; \vec{v}]. \tag{2.5}$$

This equation is called the vortex equation. It shows, in particular, that in the stationary case the field of a vortex commutes with the velocity field. Assuming, further, that $v_1 = -\Psi_y$, $v_2 = \Psi_x$, where $\Psi = \Psi(x, y, t)$ is the stream function [1], we find that r$ot \;\; \vec{v} = \vec{k}\triangle\Psi$, where $\triangle = \partial^2_{xx} + \partial^2_{yy}$ is the Laplace operator, $\vec{k}$ is the unit vector in the direction of z - axis. Setting $\Omega = \triangle\Psi$ and defining the Jacobian (or hydrodynamical bracket) by the relation

$$J(w_1, w_2) = \frac{\partial(w_1, w_2)}{\partial x, \partial y} \equiv \{w_1, w_2\}_H = w_{1x}w_{2y} - w_{1y}w_{2x}, \tag{2.6}$$

[1] Given the velocity field, it is always possible to restore the stream function:

$$\Psi(x, y, t) = \int_{(x_0, y_0)}^{(x,y)} v_2 dx - v_1 dy,$$

where $(x_0, y_0) \in \mathbb{D}$ is a reference point, and the value of the integral does not depend on the contour of integration.

we obtain the following nonlinear equation for vorticity Ω [2]:

$$\Omega_t = \{\Omega, \Psi\}_H. \tag{2.7}$$

It can be checked by direct calculation that it is possible to re-write this equation in the following hamiltonian form [9],

$$\frac{\partial \Omega}{\partial t} = \{\Omega, \mathcal{H}\}_{PB}, \quad \mathcal{H} = \frac{1}{2}\int (\nabla \Psi)^2 dx\, dy, \tag{2.8}$$

where $\mathcal{H}$ is the Hamiltonian, and the Poisson bracket on smooth functionals is defined by

$$\{F_1, G_1\}_{PB} = \int \Omega \frac{\partial(\delta F_1/\delta\Omega, \delta G_1/\delta\Omega)}{\partial(x, y)} dx\, dy. \tag{2.9}$$

To state the problem in a closed form, it is now necessary to restore the stream function from vorticity. The corresponding equation is reduced to an interior Dirichlet problem for the Poisson equation (F_0 is a given function),

$$\triangle \Psi = \Omega(x, y, t), \qquad \Psi_{|\partial\mathbb{D}} = F_0(l). \tag{2.10}$$

Its solution has the form

$$\Psi(x, y, t) = \triangle^{-1}\Omega = \int\int_{\mathbb{D}} G_0(x - x', y - y')\Omega(x', y', t)dx'dy' + \int_{\partial\mathbb{D}} \frac{\partial G_0}{\partial \vec{n}} F_0(l)dl, \tag{2.11}$$

where $G_0(x, y) = (1/2\pi)\ln r + g_0(x, y)$ is the Green function for the Dirichlet problem, $r = \sqrt{x^2 + y^2}$, and $g_0(x, y)$ is a harmonic function such that $G_0(x, y)_{|\partial\mathbb{D}} = 0$. Notice that the function Ψ in (2.10) is defined up to addition of an arbitrary harmonic function respecting the boundary condition, and satisfies certain additional relation to be discussed below in detail.

To conclude this section, we single out two special cases of the equation (2.7). The first (stationary case) is when the quantities Ω and Ψ are time - independent. We then have from (2.7): $\Omega = F_0(\Psi)$, where $F_0(\Psi)$ is an arbitrary function, and

$$\triangle \Psi = F_0(\Psi), \tag{2.12}$$

that is, we obtain an equation of elliptic type in dimension (2+0). Most interesting are the completely integrable cases of the equation (2.12) with functions $F_0(\Psi)$ of the form $\pm e^{\pm\Psi}$, $\pm\cosh\Psi$, $\pm\sinh\Psi$, $\pm\sin\Psi$, $e^{\Psi} - e^{-2\Psi}$ (elliptic version of the Tciceica's equation) and some others, which allow to apply the inverse scattering method. A range of such cases, with applications to various physical phenomena, is considered in literature (see, for instance, [10-12]). The methods used for solving (2.12) include the Hirota method, Darboux and Backlund transformations [13-15], the finite-gap integration [14], and the inverse scattering transform both for the problem on the whole plane $(x,\ y)$ [16] and on the half-plane $\{(x,\ y) : y > 0\}$[17].

Another important nontrivial partial case of the equation (2.7) arises as follows. Assume that $\Omega = \triangle\Psi = F_1(\Psi) + g$ with a certain function F_1, and $g = g(x, y, t)$. Assume further that either $F_{1\Psi} = 0$, or $\Psi_t = 0$. We then obtain from (2.7),

[2] it is easy to see that it can be re-written in terms of the stream function alone.

$$g_t = \Psi_y g_x - \Psi_x g_y. \tag{2.13}$$

. This equation, which is given, for instance, in [18], we shall call the simplified vortex equation. As well as (2.7), this equation is of significant physical interest, and, importantly, is completely integrable.

3. EXACT SOLUTIONS OF EQUATION (2.13).

In this section we construct an exact solution of the simplified vortex equation (2.13) assuming that $g = \triangle\Psi + f(x, y, t)$, where $f(x, y, t)$ is a known function.

It is easy to verify by straightforward computation that the equation (2.13) can be represented as a compatibility condition for the following system of scalar equations [3]:

$$(g - \lambda)\varphi = 0, \quad \varphi_t = \Psi_y \varphi_x - \Psi_x \varphi_y, \tag{3.1}$$

where $\varphi = \varphi(x, y, t)$ is a complex-valued function and $\lambda \in \mathbb{C}$ is the spectral parameter. It is convenient for our purposes to put the first equation of the pair in a somewhat different form to obtain,

$$g_x \varphi_y = g_y \varphi_x, \quad \varphi_t = \Psi_y \varphi_x - \Psi_x \varphi_y. \tag{3.2}$$

The existence of such representation allows us to construct a family of exact solutions of the equation (2.13). To this end, we use the method of the generalized Darboux transformation [4] (see, for example, [19]).

Let

$$\tilde{\varphi}(x, y, t) = \frac{Q(\varphi, \varphi_1)}{\varphi_1(x, y, t)}, \qquad Q(\varphi, \varphi_1) = \int_{(x_0, y_0, t)}^{(x, y, t)} q(\varphi, \varphi_1), \tag{3.3}$$

where $q(\varphi, \varphi_1) = (\varphi_x \varphi_1 - \varphi \varphi_{1x})dx + (\varphi_y \varphi_1 - \varphi_{1y}\varphi)dy$ is a 1-form, and φ_1 is a fixed solution of (3.2); then the integral $Q(\varphi, \varphi_1)$ does not depend on the contour of integration by virtue of the first equality. Let us verify the covarianse of the system (3.2) under the transformation $\varphi \to \tilde{\varphi}$, $g \to \tilde{g}$, $\Psi \to \tilde{\Psi}$. For the first equation this leads to a dressing relation of the form

$$\tilde{g}_y = \tilde{g}_x \frac{\varphi_{1y}}{\varphi_{1x}}, \tag{3.4}$$

which implies that[5]

$$\tilde{g}(x, y, t) = f_0(\varphi_1(x, y, t)), \tag{3.5}$$

where $f_0(\varphi_1(...))$ is an arbitrary function. The verification of the second relation gives

[3] In the present form this Lax pair was suggested by A.Ja.Kazakov.

[4] This method is also known as Moutard's transform.

[5] For a first order partial differential equation of the form $\{Y, A\}_H = 0$, with a given $A = A(x, y)$, the function $Y = R_0(A(x, y))$ is a solution for any function $R_0(.)$.

$$Q_t - Q_x\tilde{\Psi}_y + Q_y\tilde{\Psi}_x + Q[-(\ln\varphi_1)_t + \tilde{\Psi}_y(\ln\varphi_1)_x - \bar{\Psi}_x(\ln\varphi_1)_y] = 0. \tag{3.6}$$

Solving this linear first order partial differential equation (recall that the function $\tilde{\Psi}$ is known), we find Q explicitly.

Taking (3.5) into account, we have from (2.11),

$$\tilde{\Psi}(x,y,t) = \int\int_D G_0(x-x',y-y',t)[f_0(\varphi_1(x',y',t)) - f(x',y',t)]dx'dy', \tag{3.7}$$

which solves the initial problem.

Clearly, the procedure suggested can be repeated and one can construct the N-dressed solutions, but we shall not dwell on that in detail here.

4. SOLUTION OF THE VORTEX EQUATION.

1. *Linearized equation.*

Let us first consider the linearization of the equation (2.7). Let $\Psi(x,y,t) = \Psi_1(x,y,t) + \Psi_0(x,y,t)$, $\Omega(x,y,t) = \Omega_1(x,y,t) + \Omega_0(x,y,t)$, where $\Omega_0 = \triangle\Psi_0 = 0$, Ψ_0 is a known solution of (2.7). We then have

$$(\triangle\Psi_1)_t = (\triangle\Psi_1)_x\Psi_{0y} - (\triangle\Psi_1)_y\Psi_{0x}. \tag{4.1}$$

Taking $\triangle\Psi_1$ in the form

$$\triangle\Psi_1 \sim a(\hat{\alpha},\ \hat{\beta},\ \hat{\omega})\exp\{i(\hat{\alpha}x + \hat{\beta}y + \hat{\omega}t)\}, \quad \text{where} \quad a(\hat{\alpha},\ \hat{\beta},\ \hat{\omega}) = \bar{a}(-\hat{\alpha},\ -\hat{\beta},\ -\hat{\omega}), \tag{4.2}$$

and substituting it in (2.7), we obtain the following dispersion relation for the waves (assuming that $\Psi_{0x} = p_0$, $\Psi_{0y} = q_0$, where p_0, q_0 are constants)

$$\hat{\omega} = p_0\hat{\alpha} - q_0\hat{\beta}. \tag{4.3}$$

2. *Certain partial solutions.*

Using various approaches, we now construct a set of partial solutions of (2.7).

a). Let $\Psi(x,y,t) = (X(x) + Y(y))T(t)$. We then have from (2.7):

$$a(X_{1x} + Y_{1y}) = X_{1xx}Y_1 - Y_{1yy}X_1,$$

where $X_1 = X_x, Y_1 = Y_y, a = T_t/t^2 = const$. Searching for the functions X_1, Y_1 polynomial in x and y respectively, we find,

$$\Psi(x,\ y,\ t) = \left[c(x^2 - y^2) + d_1x + d_2y + h\right]\frac{a}{t}, \quad \Omega = \triangle\Psi = 0, \tag{4.4}$$

where $t > 0$, and c, d_1, d_2, h are constants. This solution is quadratic in the spatial variables and slowly (sub- exponentially) decreases in time. Obviously, it describes a vortex-free flow of the fluid.

b). Let $\Psi = X(x) + Y(y) + T(x,y,t)$. We then have,

$$(T_{xx} + T_{yy})_t = (X_{xx} + T_{xx} + T_{yy})_x(Y_y + T_y) - (Y_{yy} + T_{xx} + T_{yy})_y(X_x + T_x).$$

Letting $X = a_1x^2 + b_1x + c_1$, $Y = a_2y^2 + b_2y + c_2$, $T = c\cos(\alpha x + \beta y + \gamma t + \delta)$, where a_i, b_i, c_i, $i = 1, 2$, c, α, β, γ, δ are constants, we obtain: $a_1 = a_2 = 0$,

$$\begin{gathered}\Psi(x,y,t) = b_1x + b_2y + d + c\cos(\alpha x + \beta y + \gamma t + \delta), \\ \Omega = -c(\alpha^2 + \beta^2)\cos(\alpha x + \beta y + \gamma t + \delta),\end{gathered} \tag{4.5}$$

and $\gamma = \alpha b_2 - \beta b_1$. These formulae were used, in particular, in [20] to construct solutions on the basis of group-theoretical considerations and describe a stream function linear in spatial variables on the background of spatial-temporal oscillations.

In a similar way, one can obtain a solution growing in the absolute value in all variables:

$$\begin{gathered}\Psi(x,y,t) = b_1x + b_2y + d + c\cosh(\alpha x + \beta y + \gamma t + \delta), \\ \Omega = c(\alpha^2 + \beta^2)\cosh(\alpha x + \beta y + \gamma t + \delta),\end{gathered} \tag{4.6}$$

where $\gamma = \alpha b_2 - \beta b_1$.

c). Let us define the complex variables $z = x + iy$, $\bar{z} = x - iy$, and let $\partial = (1/2)(\partial_x - i\partial_y)$, $\bar{\partial} = (1/2)(\partial_x + i\partial_y)$. On passing to these variables in (2.7), we obtain

$$\Psi_{z\bar{z}t} = 2i(\Psi_{z\bar{z}\bar{z}}\Psi_z - \Psi_{zz\bar{z}}\Psi_{\bar{z}}). \tag{4.7}$$

Now, setting $\chi = \Psi_{\bar{z}}$, we have,

$$\chi_{zt} = 2i(\chi_{z\bar{z}}\bar{\chi} - \chi_{zz}\chi). \tag{4.8}$$

Assume that $\chi_{\bar{z}} = 0$, that is, χ is an analytic function in the domain $\mathbb{D}$. Then it satisfies the following nonlinear equation,

$$\chi_{zt} = -2i\chi_{zz}\chi. \tag{4.9}$$

In the simplest case when $\chi(z,\ t) = Z(z)T(t)$, this gives,

$$\frac{T_t}{T^2} \equiv a^2, \quad \mathrm{E}i\ (\ln Z) = \mathrm{E}i\ (\ln Z_0) + \frac{a^2}{2i}e^{\frac{C_1}{a^2}}(z - z_0), \tag{4.10}$$

where $\mathrm{E}i\ (u) = \int_{-\infty}^{u}(e^t/t)dt$ is the integral exponential function, a, C_1 are arbitrary constants, and $Z_0 = Z(z_0)$. Formally, one can write,

$$Z(z) = e^{\mathrm{E}i^{-1}[\ln Z_0 + \frac{a^2}{2i}e^{\frac{C_1}{a^2}}(z-z_0)]}. \tag{4.11}$$

Another solution of (4.9) is found if we plug χ in the form $\chi(z,t) = \chi(\xi)$, where $\xi = \alpha z + \beta t + \gamma$, α, β, γ being constants. The solution is then given by

$$\chi(z,t) = \xi. \tag{4.12}$$

In both cases (4.10) - (4.11) and (4.12), using the Green-Cauchy representation (the $\bar{\partial}$ -problem relation) [21], we obtain for the function Ψ in the case of a bounded domain $\mathbb{D}$,

$$\Psi(z,\bar{z},t) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\chi_0(\zeta,\bar{\zeta},t)}{\zeta - z} d\zeta + \frac{1}{2\pi i}\int\int_{\mathbb{D}} \frac{\chi(\zeta,\bar{\zeta},t)}{\zeta - z} d\zeta \wedge d\bar{\zeta}, \tag{4.13}$$

where the domain $\mathbb{D}$ is assumed bounded, and $\partial\mathbb{D}$ stands for its boundary, $\chi_0(z,\bar{z},t) = \Psi(z,\bar{z},t)_{|z\in\partial\mathbb{D}}$.

2. *Lax representation and the method of Darboux transformation.*

The equation (2.7) can be represented as a compatibility condition for the following over-determined system of scalar equations [4, 22] [6]:

$$L\varphi = \lambda\varphi, \quad \varphi_t = -A\varphi, \tag{4.14}$$

where $L\varphi = \{\Omega,\varphi\}_H$, $A\varphi = \{\Psi,\varphi\}_H$, $\lambda \in \mathbb{C}$ is the spectral parameter, and the symbol $\{..\}_H$ refers to the "hydrodynamical" bracket defined by (2.6). Here the compatibility, and hence the integrability of (2.7), is understood in a "weak" sense, that is, under the additional condition

$$\{\Omega_t + \{\Psi,\Omega\}_H, \varphi\}_H = 0. \tag{4.15}$$

We now apply the method of Darboux transformation to construct a solution of the equation (2.7). Two different versions of the method will be considered.

i). We use a matrix transform. It is easy to notice that the system (4.14) can be rewritten in a matrix form (taking into account that $\Psi = \bar{\Psi}$, $\Omega = \bar{\Omega}$),

$$\Omega_x\Phi_y - \Omega_y\Phi_x = \Phi\Lambda, \qquad \Phi_t = \Psi_y\Phi_x - \Psi_x\Phi_y, \tag{4.16}$$

where

$$\Phi = \Phi(x,y,t,\lambda) = \begin{pmatrix} 0 & \varphi \\ \bar{\varphi} & 0 \end{pmatrix}, \qquad \Lambda = diag(\bar{\lambda},\ \lambda),$$

and the condition (4.15) is assumed to hold.

Let us check the covariance of the equations (4.16), $\Phi \to \tilde{\Phi}$, $\Psi \to \tilde{\Psi}$, $\Omega \to \tilde{\Omega}$, under the transformation

$$\tilde{\Phi} = \Phi - \sigma_1\Phi\Lambda, \quad \sigma_1 = \Phi_1\Lambda_1\Phi_1^{-1}, \tag{4.17}$$

where $\Phi_1 = \Phi(x,y,t,\lambda_1)$ is a solution of the system (4.16) corresponding to a fixed value $\lambda = \lambda_1$. We then arrive at the following "dressing" relations,

$$\tilde{\Psi}_x = \Psi_x,\ \ \tilde{\Psi}_y = \Psi_y,\ \ \{\tilde{\Omega}_x,\Omega\}_H = 0,\ \ \{\tilde{\Omega},\sigma_1\}_H = 0,\ \ \tilde{\Omega}_x = \Omega_x, \tag{4.18}$$

from which we infer that

[6] From the viewpoint of the spectral theory the first equation in (4.14) can be interpreted as an eigenvalue problem for the operator of "generalized" rotation of the plane, $\Omega_x\frac{\partial}{\partial y} - \Omega_y\frac{\partial}{\partial x}$, which is easily shown (see, for instance, [2]) to be antisymmetric.

$$\tilde{\Psi} = \Psi + F_1, \quad \tilde{\Omega} = \Omega + G_1, \tag{4.19}$$

where $F_1 = F_1(t)$, $G_1 = G_1(t)$ are arbitrary functions of time.

Notice, further, that the substitution $\Psi \to \Psi + F_1(t) + h(x, y, t)$, $\Omega \to \Omega + G_1(t)$,

$$h = h(x, y, t) = \sum_{i=1}^{L} d_i(t) h_i(x, y), \tag{4.20}$$

where the integer L and the functions $d_i(t)$, $i = 1, L$ are arbitrary, and $h_i(x, y)$ are harmonic, does not change the form of the equation (2.7) if the condition [7]

$$G_{1t} = \{\Omega, \sum_{i=1}^{L} d_i(t) h_i\}_H, \tag{4.21}$$

is satisfied.

This allows to write down the new solution of the equation (2.7) as follows,

$$\tilde{\Psi}(x, y, t) = \Psi(x, y, t) + F_1(t) + h(x, y, t). \tag{4.22}$$

ii). We now provide an alternative Darboux transformation, which apparently makes it possible to construct a wider class of solutions, as compared to the above. To this end, we check the co-invariance of the system (4.14) under the transformation $\varphi \to \tilde{\varphi}$, $\Psi \to \tilde{\Psi}$, $\Omega \to \tilde{\Omega}$, where

$$\tilde{\varphi} = T(x, y, t)(\varphi_x - \sigma_1 \varphi), \tag{4.23}$$

$\sigma_1 = (\ln \varphi_1)_x$, φ_1 is a fixed solution of (4.14) corresponding to $\lambda = \lambda_1$, and $T = T(x, y, t)$ is a function to be determined later [8]. Matching the coefficients at φ, φ_x and φ_{xx}, we then obtain, after a cumbersome calculation, the following over-determined system of dressing relations,

$$\{\tilde{\Omega}, \Omega\}_H = 0, \quad \{\tilde{\Psi} - \Psi, \Omega\}_H = 0, \tag{4.24a}$$

$$\lambda T[\sigma_1 - \frac{\tilde{\Omega}_x}{\Omega_x}(\frac{\Omega_{xx}}{\Omega_x} + \sigma_1)] - \{\tilde{\Omega}, T\sigma_1\}_H = 0, \tag{4.24b}$$

$$\{\tilde{\Omega}, \ln T\}_H + \tilde{\Omega}_x[(\frac{\Omega_y}{\Omega_x})_x - \sigma_1 \frac{\Omega_y}{\Omega_x}] + \tilde{\Omega}_y \sigma_1 + \lambda(\frac{\tilde{\Omega}_x}{\Omega_x} - 1) = 0, \tag{4.24c}$$

$$(T\sigma_1)_t + \{\tilde{\Psi}, T\sigma_1\}_H - \lambda T[\frac{\Psi_x - \tilde{\Psi}_x}{\Omega_x^2}(\Omega_x \sigma_1 + \Omega_{xx}) - \frac{\Psi_{xx}}{\Omega_x}] = 0, \tag{4.24d}$$

$$\begin{aligned}(\ln T)_t + \tilde{\Psi}_x[(\ln T)_y + (\frac{\Omega_y}{\Omega_x})_x - \sigma_1 \frac{\Omega_y}{\Omega_x}] + \tilde{\Psi}_y[-(\ln T)_x + \sigma_1] + \\ +[\Psi_{yx} - \Psi_x(\frac{\Omega_y}{\Omega_x})_x - \Psi_{xx}\frac{\Omega_y}{\Omega_x} - \frac{\sigma_1}{\Omega_x}\{\Omega, \Psi\}_H] + \frac{\lambda}{\Omega_x}(\tilde{\Psi}_x - \Psi_x) = 0.\end{aligned} \tag{4.24e}$$

[7] This remark can be generalized to the case $h(x, y, t) = \sum_{i=1}^{L} h_i(x, y, t)$, $\triangle h = \triangle h_i = 0$, $i = 1, L$.

[8] In paper [22] the transformation (4.23) was used in the case $\lambda = 0$, $T(x, y, t) = 1/\Omega_x$.

Let us analyze the compatibility of this system, limiting ourselves to simpler reductions. Obviously, there are two essentially different cases: $\lambda = 0$ and $\lambda \neq 0$, when the coefficients at λ should vanish.

Let $\lambda \neq 0$. We then infer that $\tilde{\Psi}_x = \Psi_x$ from (4.24e); $\Psi_{xx} = 0$ from (4.24d); $\tilde{\Omega}_x/\Omega_x = 1$ from (4.24c). On taking into account (4.24a), we obtain that

$$\tilde{\Psi} = [\alpha_0(y,t) + c_1]x + c_2,$$

where $\alpha_0(,)$ is an arbitrary function, and c_1, c_2 are arbitrary constants. From this we find that $\tilde{\Omega} = \alpha_{0yy}x$, $\tilde{\Omega}_y = \alpha_{0yyy}x$, $\tilde{\Omega}_x = \alpha_{0yy}$, and that $\Omega_y/\Omega_x = (\ln \alpha_{0yy})_y x$ in view of the first equality in (4.2a). Eventually, we arrive at a rather complicated over-determined system of equations on the function α_0, which is likely to be incompatible.

Now let $\lambda = 0$, $T = 1$. On calculating, we then obtain that (4.24) transforms into a system of the form

$$\{\tilde{\Omega}, \Omega\}_H = 0, \ \{\tilde{\Omega}, \sigma_1\}_H = 0, \ \{\tilde{\Psi} - \Psi, \Omega\}_H = 0, \ \{\tilde{\Omega}, \Psi_x\}_H = 0,$$

$$(\frac{\Omega_y}{\Omega_x})_x = 0, \quad \sigma_{1t} + \{\tilde{\Psi}, \sigma_1\}_H = 0. \tag{4.25}$$

Substituting

$$\tilde{\Psi} = \Psi + U_1, \ \tilde{\Omega} = \Omega + V_1, \quad V_1 = \triangle U_1, \tag{4.26}$$

in (4.25), we get the following system of equations on U_1:

$$\{\Omega + V_1, \Omega\}_H = 0, \ \{\Omega + V_1, \sigma_1\}_H = 0, \ \{\Omega + V_1, \Psi_x + U_{1x}\}_H = 0,$$

$$\{U_1, \Omega\}_H = 0, \quad \sigma_{1t} + \{\Psi + U_1, \sigma_1\}_H = 0. \tag{4.27}$$

The compatibility of the first three equations becomes obvious if we set $\Omega + V_1 = p_0(\Omega, \sigma_1, \Psi_x + U_{1x})$, where $p_0(.)$ is an arbitrary function. It then follows from the first relation that $\{\triangle U_1, \Omega\}_H = 0$, that is, $\triangle U_1 = q_0(\Omega)$ where $q_0(.)$ is an arbitrary function, hence $U_1 = \triangle^{-1} q_0(\Omega) \equiv g_0(\Omega)$. This means that the fourth equation in (4.27) is compatible with the first three. Moreover, the fourth equation in (4.27) implies that $U_{1y}/U_{1x} = \Omega_y/\Omega_x$, and the requirement of the compatibility with the remaining equation gives

$$U_1(x,y,t,\lambda_1) = -\int_{-\infty}^{x} \frac{\sigma_{1t} + \{\Psi, \sigma_1\}_H}{\{\Omega, \sigma_1\}_H} dx \tag{4.28}$$

under the assumption that $U_1 = U_1(x,y,t,\lambda_1)$, $U_1(-\infty,y,t,\lambda_1) = 0$ and $\{\Omega, \sigma_1\}_H \neq 0$.

Thus, we have obtained dressing relations allowing to construct a new explicit solution of the equation (2.7) from a known one. It is also possible to obtain the corresponding multiple dressing formulae in the spirit of the Darboux transform method [23].

5. CONCLUSION

Several procedures for constructing the solutions of the $2D$ vortex equation have been suggested in this paper. A number of questions remains open, however: the meaning of the notion of " weak integrability", the possibility of application of the multidimensional inverse scattering transform method, the Hamiltonian formalism, and, especially, quantization of the vortices, to mention a few.

Notice that various extended versions of this equation are of doubtless interest. For instance, the equation for the so-called Rossby waves [6, 7] has the form

$$\Omega_t = \{\Omega, \Psi\}_H - \beta\Omega,$$

where β is a parameter, that is, it differs from (2.7) by an additional term. As shown in [4], it also admits a Lax representation (which is easily recovered from (4.14)), hence exact solutions can be constructed.

Let us emphasize that there is a wide range of adjacent topics related to the theme of this paper: the problem of decay of a turbulent spell, models of vortex chains, behaviour of vortex ensembles in geophysics, plasma physics, and many others. These reasons urge a further analysis of the equation (2.7) and its generalizations.

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